derivatives of their viscosity are comparable to the analogous values for liquids in which intermolecular hydrogen bonds are absent.

NOTATION

P, pressure; T, temperature; N, Avogadro's number; R, universal gas constant; V, molar volume; h, Planck's constant; η and η_0, dynamic viscosity of liquid at elevated and atmospheric pressure, respectively; ρ and ρ_0, density of liquid at elevated and atmospheric pressure, respectively; K_0, specific volume of "incompressible" nuclei; $\varkappa$, linear temperature coefficient of specific volume of incompressible nuclei; $\Delta\Phi_\eta^{\neq}$, free activation energy of viscous flow; $\Delta H_\eta^{\neq}$, activation enthalpy; $\Delta V_\eta^{\neq}$, activation volume; α, isobaric coefficient of cubical expansion.

LITERATURE CITED

1. A. M. Sukhotin, A. F. Kazankina, L. V. Zaretskaya, et al., "Nonflammable nontoxic liquids for hydraulic systems," Khim. Tekhnol. Topl. Masel, No. 7, 55-58 (1977).
2. A. M. Sukhotin (editor), Nonflammable Heat-Transfer Agents and Hydraulic Fluids (Handbook) [in Russian], Khimiya, Leningrad (1979).
3. J. Pimentel and O. MacKellan, Hydrogen Bond [Russian translation], Mir, Moscow (1964).
4. Yu. A. Atanov, D. I. Kuznetsov, and A. A. Seifer, "Viscous flow of liquid polysiloxanes at high pressures," Vysokomol. Soedin., A15, 680-684 (1973).
5. S. Glasstone, K. Laidler, and H. Eyring, Theory of Absolute Reaction Rates [Russian translation], PPR, Moscow (1948).
6. V. N. Belonenko, "Effect of state parameters on rheological and thermophysical properties of oils of different composition," Author's Abstract of Candidate's Dissertation, Engineering Sciences, Moscow (1979).
7. P. W. Bridgman, Physics of High Pressures [Russian translation], ONTI, Moscow—Leningrad (1935), pp. 335-360.

VARIOUS CASES OF MUTUAL DISPLACEMENT OF IMMISCIBLE FLUIDS IN POROUS MEDIA

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All possible solutions to the problem of mutual displacement of immiscible fluids are obtained for the case of one-dimensional filtration with piecewise-uniform initial conditions.

Displacement of crude oil or natural gas from a porous medium by water plays an important role in oil and gas extraction. This process is basic in application of secondary methods of oil extraction as well as in the tapping of underground basins formed in water-bearing bedrocks.

For evaluating the effectiveness with which crude oil or natural gas is displaced by an immiscible fluid, one must know how the saturation of one of the phases in the bedrock varies. During the displacement process there forms a zone of concurrent flow of both fluids, the displacing fluid flowing through some pores and the displaced fluid flowing through others. Accordingly, the displacement of immiscible fluids can be treated as a process of two-phase filtration.

In the case of uniform flow of incompressible fluids, when the surface tension between the two phases is small and the capillary pressure as well as the effects of gravity are negligible, the displacement process is amenable to a simple mathematical description [1] such as that given by Buckley and Leverett. In this formulation, the problem was subsequently analyzed by several authors. In one study [2], the general properties of the saturation

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I. M. Gubkin Moscow Institute of the Petrochemical and Gas Industry. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 41, No. 4, pp. 656-662, October, 1981. Original article submitted July 9, 1980.

field during displacement were analyzed, in other studies were proposed numerical methods of calculating a nonuniform flow [3]. However, no complete study of various cases of displacement by immiscible fluids has ever been made. In this study all possible solutions to the problem will be shown, for piecewise-uniform initial conditions, which are of fundamental importance in practical applications.

We will consider the displacement of fluid 2 from a bedrock by fluid 1 which is pumped in. We assume that both fluids are incompressible and disregard capillary as well as gravitational forces. The zone of two-phase filtration forming in the process can be described by equations which in the one-dimensional case reduce to a single quasilinear differential equation for the saturation σ of the displacing phase [2]

$$m\frac{\partial\sigma}{\partial t}+\frac{q(t)}{x^{\nu-1}}\frac{\partial f(\sigma)}{\partial x}=0\,, \tag{1}$$

where ν = 1, 2, 3 for, respectively, forward, radial, or spherical flow. The function $f(\sigma)$ here is defined as

$$f(\sigma)=\frac{k_1(\sigma)}{k_1(\sigma)+\mu_0 k_2(\sigma)}\,,\quad \mu_0=\mu_1/\mu_2\,,$$

and is equal to the fraction of the displacing fluid 1 in the total stream. Numerous experiments have established [4] that the relative penetration factors $k_1(\sigma)$ and $k_2(\sigma)$ do not depend on the ratio of the viscosity coefficients but are determined by the structure of the porous medium.

The function $f(\sigma)$, called the Buckley–Leverett function, determines the degree to which the displacing fluid replaces the displaced one and also the pattern of the saturation distribution over the bedrock. This function represents the ratio of filtration rates, viz., filtration rate of the displacing fluid to resultant filtration rate. The problems of enhancing the oil or gas extraction reduce largely to application of measures affecting the bedrock so that the function $f(\sigma)$ will change in the direction of a more nearly complete displacement.

As the degree of saturation increases, function $f(\sigma)$ increases monotonically from 0 to 1. The graph of this function contains a characteristic inflection point σ_i with a concave segment and a convex segment, where the derivative $f''(\sigma)$ is, respectively, larger and smaller than zero [1]. The latter feature, as will become evident later on, distinguishes this problem from those in gasdynamics.

It is noteworthy that in the analysis of bidirectional processes occurring, e.g., during cyclic tapping of underground gas reserves, extraction of the fluid from the bedrock is described by the same Eq. (1), with an appropriately different reference system and with function $f(\sigma)$ replaced by function $f_1(\sigma) = 1 - f(1 - \sigma)$ determining the displacement of fluid 1 by fluid 2.

For convenience, we introduce the new independent variables

$$\tau=\int_0^t\frac{q(t)}{m}\,dt\,,\quad \xi=\frac{x^\nu}{\nu}\,,\quad \nu=1,2,3\,.$$

The quantity ξ can be regarded as denoting the volume of a flow tube between the initial section and section x, the quantity τ can be regarded as denoting the volume of fluid pumped into the bedrock during the time t. The fundamantal Eq. (1) then becomes

$$\frac{\partial\sigma}{\partial\tau}+f'(\sigma)\frac{\partial\sigma}{\partial\xi}=0\,. \tag{2}$$

One must add here the initial and boundary conditions

$$\sigma(\xi,0)=\sigma_0(\xi)\,;\quad \sigma(0,\tau)=\sigma^0(\tau)\,.$$

We will consider the self-adjoint solutions which correspond to the conditions

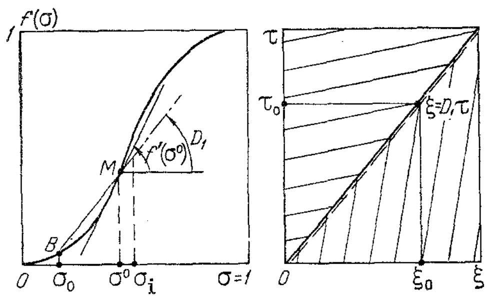

Fig. 1. Constructed solution and field of characteristics for the case $0 \leqslant \sigma_0 \leqslant \sigma^0 < \sigma_i$.

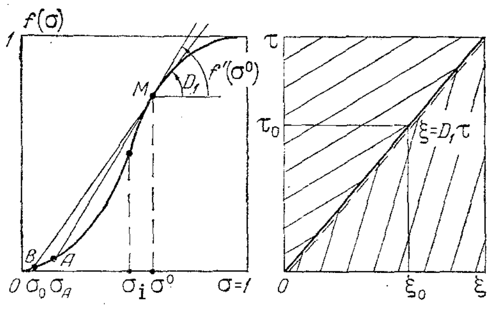

Fig. 2. Constructed solution and field of characteristics for the case $0 \leqslant \sigma_0 < \sigma_A < \sigma_i \leqslant \sigma^0$.

$$\sigma^0(\tau) = \sigma^0 = \text{const when } \xi = 0,$$
$$\sigma_0(\xi) = \sigma_0 = \text{const when } \tau = 0 \text{ и } \xi > 0, \tag{3}$$

where σ^0 and σ_0 are arbitrary constants and, without loss of generality, $\sigma^0 > \sigma_0$. The analogous problem for $\sigma^0 = 1$ was considered in another study [5].

Upon introduction of the new variable $\zeta = \xi/\tau$, the problem reduces to the ordinary differential equation

$$[\zeta - f'(\sigma)]\frac{d\sigma}{d\zeta} = 0 \tag{4}$$

for the function $\sigma = \sigma(\zeta)$, $0 \leqslant \zeta \leqslant \infty$, satisfying the conditions $\sigma(0) = \sigma^0$ and $\sigma(\infty) = \sigma_0$. Equation (4) has two sets of solutions:

$$1)\ \xi/\tau = f'(\sigma), \tag{5}$$
$$2)\ \sigma = \text{const}. \tag{6}$$

A solution (5) is given in an implicit form. It follows from the properties of function $f'(\sigma)$ that it satisfies the first condition when $\zeta = 0$ and generally does not satisfy the second condition. A solution (6), with the constant equal to σ_0, will conversely satisfy the condition at infinity but not the first boundary condition.

Thus, constructing a self-adjoint solution involves "collocating" the elementary continuous solutions (5) and (6) (centered waves and ranges of constant σ) through jumps and determining the parameters which characterize these solutions as well as the discontinuities. Inasmuch as the discontinuities and the elementary solutions (simple waves) are determined by a finite number of parameters, this problem becomes an algebraic one and it follows from its self-adjointness that the discontinuity lines are straight in the plane of variables ξ, τ, which means that the discontinuities propagate at a constant velocity D.

The characteristics of the differential equation (2) are straight lines

$$\xi = \xi_0 + \tau f'(\sigma),$$

along which the saturation σ remains constant. This means that each saturation level σ "propagates at the velocity" $d\xi/d\tau$ proportional to $f'(\sigma)$. In the case of straight-parallel motion ($\nu = 1$) $d\xi/d\tau$ is the true velocity of propagation of a given saturation level.

On the $\xi = \xi(\tau)$ line of discontinuity of the solution $\sigma(\xi, \tau)$ the relation

$$D = \dot{\xi}(\tau) = \frac{f(\sigma^+) - f(\sigma^-)}{\sigma^+ - \sigma^-} \tag{7}$$

is satisfied which follows from the integral law of mass conservation. Here the superscripts "+" and "−" refer to values of σ, respectively, to the right and to the left of the discontinuity line. Equality (7) has a simple geometrical meaning: the velocity D of a discontinuity is equal to the tangent of the angle which the chord AB connecting points on the $f(\sigma)$ curve with abscissas σ^+ and σ^- makes with the σ axis.

In order to extract the unique solution, it is necessary to satisfy the condition that the discontinuities be stable. In the given case of a sign-reversing function $f''(\sigma)$ this condition can be expressed in the form of the inequality

$$\frac{f(\sigma^*)-f(\sigma^+)}{\sigma^*-\sigma^+} \geqslant D \geqslant \frac{f(\sigma^*)-f(\sigma^-)}{\sigma^*-\sigma^-}, \tag{8}$$

which any value $\sigma^* \in (\sigma^-, \sigma^+)$ satisfies. This condition has been established originally in another study [6]. When σ^- and σ^+ belong in the interval where $f''(\sigma) > 0$, then the condition of stability can be formulated simpler as

$$f'(\sigma^+) \geqslant D \geqslant f'(\sigma^-), \tag{9}$$

i.e., that the slope of the discontinuity line must be smaller than the slope of the characteristics of Eq. (2) calculated from the saturation σ^+ "behind a discontinuity" but larger than the slope of the characteristics calculated from the saturation σ^- "before a discontinuity."

It is well known [6] that a generalized solution to the given problem exists and that it will be unique when condition (8) or, in the case of $f''(\sigma) > 0$, condition (9) has been satisfied. Let us construct all solutions corresponding to various locations of σ^0 and σ_0 relative to σ_i, the coordinate of the inflection point of the $f(\sigma)$ curve. The following situations are possible.

1. $0 \leqslant \sigma_0 \leqslant \sigma^0 < \sigma_i$ (Fig. 1). Points σ_0 and σ^0 lie on the segment where $f''(\sigma) > 0$. In this case the velocity D of a saturation discontinuity is higher than that of perturbations $f'(\sigma_0)$ before a discontinuity but lower than that of perturbations $f'(\sigma^0)$ behind a discontinuity. The pattern of characteristics corresponding to this case is shown in Fig. 1. Here condition (9) is satisfied and the sought solution is

$$\sigma(\xi, \tau) = \begin{cases} \sigma^0 \text{ when } \xi < D_1\tau, \\ \sigma_0 \text{ when } \xi > D_1\tau, \end{cases}$$
$$\text{where } D_1 = \frac{f(\sigma_0)-f(\sigma^0)}{\sigma_0-\sigma^0}. \tag{10}$$

2. $0 \leqslant \sigma_0 \leqslant \sigma_i < \sigma^0$ (Fig. 2). The initial saturation and the saturation of the displacing fluid lie on opposite sides of the abscissa σ_i of the inflection point on the $f(\sigma)$ curve. Here are possible two subcases. In order to describe them, we drawn a tangent to the $f(\sigma)$ curve at the point M with the abscissa $\sigma = \sigma^0$. Owing to the biconvexity of function $f(\sigma)$, this tangent can intersect the $f(\sigma)$ curve at point A or not intersect it. Let σ_A denote the abscissa of this point A, if it exists. For determining this abscissa we have the equation

$$f(\sigma^0)-f(\sigma_A) = f'(\sigma^0)(\sigma^0-\sigma_A), \quad (\sigma_A \neq \sigma^0).$$

The subcases to be considered involve satisfying, respectively, either one of the two possible inequalities

$$\text{a) } 0 \leqslant \sigma_0 < \sigma_A; \quad \text{b) } \sigma_A \leqslant \sigma_0 \leqslant \sigma_i.$$

When the said tangent does not intersect the $f(\sigma)$ curve and the abscissa of its intersection with the σ axis is on the negative side, then the solution is constructed just as in the second subcase b).

a) Let $0 \leqslant \sigma_0 < \sigma_A$ (Fig. 2). This subcase is characterized by the equivalence of both stability conditions (8) and (9). Here the solution is realized by a jumpwise transition from saturation σ_0 before a discontinuity to saturation σ^0 behind it. Such a solution is stable because, as the graph in Fig. 2 indicates, the chord BM ($\sigma_B = \sigma_0$) of the $f(\sigma)$ curve has a slope smaller than that of the tangent AM through point M but larger than that of the tangent to this curve through point B. Thus, condition (9) is satisfied. The pattern of characteristics in this subcase is shown in Fig. 2. The solution is, just as in the case 1, given by expressions (10).

b) Now let $\sigma_A \leqslant \sigma_0 \leqslant \sigma_i$ (Fig. 3). In this case a jumpwise transition from saturation σ_0 to saturation σ^0 is not possible, because condition (8) is not satisfied.

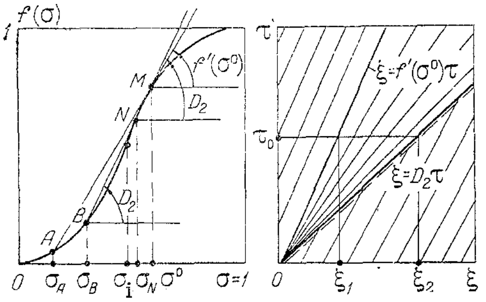

Fig. 3. Constructed solution and field of characteristics for the case $\sigma_A \leqslant \sigma_0 \leqslant \sigma_i < \sigma^0$.

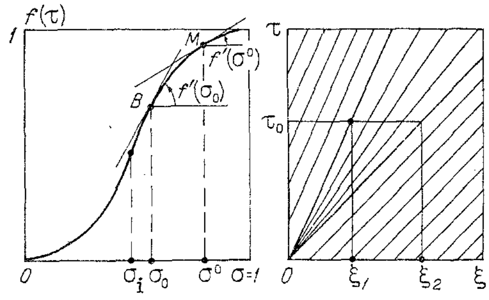

Fig. 4. Constructed solution and field of characteristics for the case $\sigma_i \leqslant \sigma_0 < \sigma^0 \leqslant 1$.

For constructing the solution, from point B ($\sigma = \sigma_0$) on the $f(\sigma)$ curve we draw the straight line BN tangent to this curve at point N ($\sigma = \sigma_N$, $\sigma_i \leqslant \sigma_N < 1$) in such a way that the slope of the chord BN will be equal to the slope of the tangent to the $f(\sigma)$ curve at point N. For determining σ_N we have the equation

$$f(\sigma_N) - f(\sigma_0) = f'(\sigma_N)(\sigma_N - \sigma_0), \quad (\sigma_N \neq \sigma_0).$$

The solution consists of a jumpwise transition from the initial saturation σ_0 to the saturation σ^0 along a discontinuity which propagates at the velocity D_2

$$D_2 = \frac{f(\sigma_0) - f(\sigma_N)}{\sigma_0 - \sigma_N},$$

of a centered rarefaction wave $\xi/\tau = f'(\sigma)$, $(\sigma_N \leqslant \sigma \leqslant \sigma^0)$, and a range of constant $\sigma = \sigma^0$ for $\xi \leqslant f'(\sigma^0)\tau$. The pattern of characteristics for this subcase is shown in Fig. 3. The constructed solution satisfies the stability requirements (8) and is

$$\sigma(\xi, \tau) = \begin{cases} \sigma^0 \text{ when } \xi \leqslant f'(\sigma^0)\tau, \\ \varphi(\xi/\tau) \text{ when } f'(\sigma^0)\tau \leqslant \xi \leqslant f'(\sigma_N)\tau, \\ \sigma_0 \text{ when } \xi > f'(\sigma_N)\tau, \end{cases} \tag{11}$$

where $\varphi(\zeta)$ is determined from the condition $\zeta = f'_\sigma[\varphi(\zeta)]$. The quantities $\xi_1 = f'(\sigma^0)\tau_0$ and $\xi_2 = f'(\sigma_N)\tau_0$ determine the boundaries of a centered rarefaction wave for $\tau = \tau_0$.

The solution to the well-known Buckley–Leverett problem corresponding to $\sigma_0 = 0$ and $\sigma^0 = 1$ we will consider as the limiting case of solution (11). It is

$$\sigma(\xi, \tau) = \begin{cases} \varphi(\xi/\tau) \text{ when } \xi < f'(\sigma_N)\tau, \\ 0 \text{ when } \xi > f'(\sigma_N)\tau, \end{cases}$$

where $f'(\sigma_N) = f(\sigma_N)/\sigma_N$ and σ_N is called the frontal saturation. This solution is obviously stable.

3. $\sigma_i \leqslant \sigma_0 < \sigma^0 \leqslant 1$ (Fig. 4). In this case a jump from σ_0 to σ^0 is impossible, since condition (8) is not satisfied. The problem has a continuous solution which determines the transition from the high saturation level σ^0 to the lower saturation level σ_0 through a centered wave $\xi/\tau = f'(\sigma)$, $\sigma_0 \leqslant \sigma \leqslant \sigma^0$. The pattern of characteristics for this case is shown in Fig. 4, where $\xi_1 = f'(\sigma^0)\tau_0$ and $\xi_2 = f'(\sigma_0)\tau_0$.

We have thus covered all possible situations.

It is to be noted that the solutions to Eq. (2) for the conditions $\sigma_0(\xi) = \sigma_0$ when $0 < \xi < \xi^*$ and $\sigma_0(\xi) = 0$ when $\xi > \xi^*$ are constructed analogously. Such conditions are of interest in the study of bidirectional filtration processes during pumping and tapping of a fluid.

NOTATION

σ, saturation of the displacing phase; σ_0, initial saturation of the displacing phase in the bedrock; σ^0, saturation of the displacing phase in a pore; σ_i, abscissa of the inflection point on the $f(\sigma)$ curve; m, porosity; q(t), resultant "specific" flow rate of fluid through a current tube; $q(t) = Q(t)/ah$ and $q(t) = Q(t)/2\pi h$, respectively, for linear and radial displacement; Q(t), volume flow rate of the phases; a and h, thickness and the width of the bedrock; μ_1 and μ_2, viscosity of the displacing fluid and of the displaced fluid, respectively; $k_1(\sigma)$ and $k_2(\sigma)$, relative penetration factors; $f(\sigma)$, Buckley—Leverett function; D, velocity of propagation of a saturation discontinuity; σ^+ and σ^-, saturation levels, respectively, to the "left" and to the "right" of a discontinuity; t, time; and x, space coordinate.

LITERATURE CITED

1. R. Collins, Flow of Fluids through Porous Materials [Russian translation], Mir, Moscow (1964).
2. G. I. Barenblatt, "Filtration of two immiscible fluids through a homogeneous porous medium," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 5, 17-26 (1971).
3. V. L. Danilov and R. M. Kats, Hydrodynamic Calculations of the Mutual Displacement of Fluids in a Porous Medium [in Russian], Nedra, Moscow (1980).
4. D. A. Éfros, Research on Filtration of Inhomogeneous Systems [in Russian], Gostoptekhizdat, Leningrad (1963).
5. V. I. Maron, "Self-adjoint solution to the Buckley—Leverett problem," in: Underground Hydrogasdynamics of Crude Oil and Natural Gas: Trans. I. M. Gubkin Moscow Institute of the Petrochemical and Natural Gas Industry [in Russian], No. 79, Nedra, Moscow (1969), pp. 45-50.
6. O. A. Oleinik, "One class of discontinuous solutions to first-order quasilinear equations," Science Lectures for Higher Schools: Physical and Mathematical Sciences, No. 3, 91-98 (1958).

NUMERICAL ANALYSIS OF TRANSVERSE STREAMLINING OF A STAGGERED BUNDLE OF TUBES

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UDC 532.517.2:532.54

The difference scheme of second-order precision [1] is applied to the analysis of transverse streamlining of coaxial circular tubes in a staggered bundle by a viscous incompressible fluid.

We consider transverse streamlining of a bundle of tubes with a circular cross section (cylinders) of the same radius R* (here and henceforth the asterisk will denote a dimensional quantity) staggered parallel in a stream of a viscous heat-conducting incompressible fluid (Fig. 1). The distances between the axes of neighboring cylinders are L* in the longitudinal direction (along the stream) and L_1* in the transverse direction (across the stream). Effects due to the finiteness of the bundle dimensions are eliminated in our calculations by considering a pair of cylinders in one of the inside rows. Such a formulation of the problem will make it possible, with finite dimensions L* and L_1*, to use the conditions of periodicity of the solution at both the entrance to and the exit from the region covered by calculations, and to disregard any possible flow asymmetry even at relatively high values of the Reynolds number.

The problem will be numerically solved by a difference approximation of the Navier—Stokes and energy equations according to the Arakawa scheme [1] of second-order precision for convective terms. The derivatives with respect to time are approximated with central differences. The region ABGHCEF of the mathematical model (Fig. 1) is bounded by the planes of symmetry AB and HC, the planes BG, CD and MN, EF in which the conditions of periodicity are satisfied, and the surfaces AF, GH of cylinders. For calculating the flow around each cylinder, we place its center at the origin of its own polar system of coordinates (r, θ)

Leningrad Institute of Mechanics. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 41, No. 4, pp. 663-668, October, 1981. Original article submitted June 30, 1980.

0022-0841/81/4104-1112$07.50